

Topological characterization of canonical Thurston obstructions

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Abstract

Let f be an obstructed Thurston map with canonical obstruction Γ_f . We prove the following generalization of Pilgrim's conjecture: if the first-return map F of a periodic component C of the topological surface obtained from the sphere by pinching the curves of Γ_f is a Thurston map then the canonical obstruction of F is empty. Using this result, we give a complete topological characterization of canonical Thurston obstructions.

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1 Introduction

Thurston's characterization theorem for branched self-covers of the topological 2-sphere \mathbb{S}^2 [DH93] gives a pure topological criterion

whether a branched cover f can be geometrized, i.e. if the topological sphere admits an invariant (in an appropriate sense) with respect to f conformal structure. The theorem states that the latter is not possible only if there exists a Thurston obstruction for f which is a collection of simple closed curves on \mathbb{S}^2 .

Pilgrim refined the proof of Thurston's theorem [Pil01] by introducing a notion of canonical Thurston obstructions and showing that a branched cover f with hyperbolic orbifold is obstructed if and only if the canonical obstruction of f is not empty. Cutting the initial branched cover f into pieces along the curves of the canonical obstruction, Pilgrim deduces Canonical Decomposition Theorem [Pil03] for branched covers. Canonical geometrization of an obstructed branched cover can now be constructed by geometrizing each piece in the canonical decomposition.

In the recent work of Bonnot, Braverman, Yampolsky [BBY10], the authors show that there exists an algorithm that can find geometrization of an unobstructed branched cover (or prove that the cover is obstructed). However, the question of algorithmically geometrizing arbitrary branched covers remains open. The first step was made in [Sel11b] where the author proved Pilgrim's conjecture from [Pil03]. We generalize the statement of Pilgrim's conjecture for all Thurston maps (also with parabolic orbifolds) and give a new prove to this conjecture (see Theorem 5.5).

It is straightforward that any curve $\gamma \in \Gamma_f$ does not intersect any other curve of any other obstruction. The converse, however, turns out to be false (but almost true). It is also immediate to see that if a minimal obstruction has the leading eigenvalue strictly greater than one, then it is a subset of the canonical obstruction. In this article, we study further properties of canonical obstructions, using the techniques developed in [Sel11b], and give their complete topological description (see Theorem 5.6).

We believe that these results will help in proving that both the canonical decomposition of a branched cover and the geometrization thereof can be found algorithmically.

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2 Basic definitions

The main setup is the same as in [DH93, Sel11b].

Let f be an orientation-preserving branched self-cover of degree $d_f \geq 2$ of the 2-dimensional topological sphere \mathbb{S}^2 . The *critical set* Ω_f is the set of all points z in \mathbb{S}^2 where the local degree of f is greater than 1. The *postcritical set* \mathcal{P}_f is the union of all forward orbits of Ω_f , i.e. $\mathcal{P}_f = \cup_{i \geq 1} f^i(\Omega_f)$. A branched cover f is called *postcritically finite* if \mathcal{P}_f is finite. More generally, a pair (f, P_f) of a branched cover $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ and a finite set $P_f \subset \mathbb{S}^2$ is called a *Thurston map* if P_f is forward invariant and contains all critical values of f (and, hence, contains \mathcal{P}_f). Denote $p_f = \#P_f$.

Two Thurston maps f and g are *Thurston equivalent* if and only if there exist two homeomorphisms $h_1, h_2: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{S}^2, P_g) \\ f \downarrow & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_2} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are homotopic relative to P_f .

A simple closed curve γ is called *essential* if every component of $\mathbb{S}^2 \setminus \gamma$ contains at least two points of P_f . We consider essential simple closed curves up to free homotopy in $\mathbb{S}^2 \setminus P_f$. A *multicurve* is a finite set of pairwise disjoint and non-homotopic essential simple closed curves. Denote by $f^{-1}(\Gamma)$ the multicurve consisting of all essential preimages of curves in Γ . A multicurve $\Gamma = (\gamma_1, \dots, \gamma_n)$ is called *invariant* if each component of $f^{-1}(\gamma_i)$ is either non-essential, or it is homotopic (in $\mathbb{S}^2 \setminus P_f$) to a curve in Γ (i.e. $f^{-1}(\Gamma) \subseteq \Gamma$). We say that Γ is *completely invariant* if $f^{-1}(\Gamma) = \Gamma$.

Every multicurve Γ has its associated *Thurston matrix* $M_\Gamma = (m_{i,j})$ with

$$m_{i,j} = \sum_{\gamma_{i,j,k}} (\deg f|_{\gamma_{i,j,k}} : \gamma_{i,j,k} \rightarrow \gamma_j)^{-1}$$

where $\gamma_{i,j,k}$ ranges through all preimages of γ_j that are homotopic to γ_i . Since all entries of M_Γ are non-negative real, the leading eigenvalue λ_Γ of M_Γ is real and non-negative (see Corollary B.2).

Remark 2.1. Note that to define the Thurston matrix M_Γ , we do not require that Γ be invariant.

A multicurve Γ is a *Thurston obstruction* if $\lambda_\Gamma \geq 1$. A Thurston obstruction Γ is *minimal* if no proper subset of Γ is itself an obstruction. We call Γ a *simple* obstruction (compare [Pil01]) if no permutation of the curves in Γ puts M_Γ in the block form

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix},$$

where the leading eigenvalue of M_{11} is less than 1. If such a permutation exists, it follows that M_{22} is a Thurston matrix of a multicurve with the same leading eigenvalue as M_Γ . It is, thus, evident that every obstruction contains a simple one.

Proposition 2.2. *A multicurve Γ is a simple obstruction if and only if there exists a vector $v > 0$ such that $M_\Gamma v \geq v$.*

Proof. Let v be a non-negative vector such that $M_\Gamma v \geq v$ with a maximal possible number of positive components. Applying a permutation if necessary, we assume that $v = (0, v_1)^T$ where $v_1 > 0$. If we write M_Γ in the corresponding block form we get:

$$M_\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}.$$

If the leading eigenvalue of M_{11} is greater than 1, then there exists a non-negative eigenvector v_2 for M_{11} such that $M_\Gamma(v_2 \ v_1)^T \geq (v_2 \ v_1)^T$. The choice of $v = (0, v_1)^T$ implies that either v is positive, or the leading eigenvalue of M_{11} is less than 1, and Γ is not a simple obstruction.

On the other hand, it is clear that if there exists a positive vector v with $Mv \geq v$, then the leading eigenvalue of M is at least 1. Therefore, if there exists a positive vector v with $M_\Gamma v \geq v$, then M_Γ can not be written in a block form as above so that the leading eigenvalue of M_{11} is less than 1. \square

Note that every minimal obstruction is simple, and that a union of two disjoint simple obstructions is also simple. A *Levy cycle* is a multicurve $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ such that for every $i = \overline{1, n}$, there exists a preimage component of γ_i that is homotopic to γ_{i+1} and is

mapped to γ_i by f with degree 1 (we set $\gamma_{n+1} = \gamma_1$). Every Levy cycle is a Thurston obstruction.

Thurston's original characterization theorem is formulated as follows:

Theorem 2.3 (Thurston's Theorem [DH93]). *A postcritically finite branched cover $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with hyperbolic orbifold is either Thurston-equivalent to a rational map g (which is then necessarily unique up to conjugation by a Moebius transformation), or f has a Thurston obstruction.*

Remark 2.4. In the original formulation in [DH93], a Thurston obstruction was required to be invariant. Ommiting this requirement makes the statement of the theorem weaker in one direction and stronger in the other direction. However, in [Sel11a] we showed that if there exists a Thurston obstruction for f , then there also exists a simple invariant obstruction (see Propostion 3.2).

General rigorous definition of orbifolds and their Euler characteristic can be found in [Mil06a]. In our case, there is a unique and straightforward way to construct the minimal function v_f of all functions $v: \mathbb{S}^2 \rightarrow \mathbb{N} \cup \{\infty\}$ satisfying the following two conditions:

- (i) $v(x) = 1$ when $x \notin P_f$;
- (ii) $v(x)$ is divisible by $v(y) \deg_y f$ for all $y \in f^{-1}(x)$.

We say that f has hyperbolic orbifold $O_f = (\mathbb{S}^2, v_f)$ if the Euler characteristic of O_f

$$\chi(O_f) = 2 - \sum_{x \in P_f} \left(1 - \frac{1}{v_f(x)} \right) \quad (1)$$

is less than 0, and parabolic orbifold otherwise. We discuss Thurston maps with parabolic orbifolds in more detail in Section 4.

3 Thurston's pullback map and canonical obstructions

Let \mathcal{T}_f be the Teichmüller space modeled on the marked surface (\mathbb{S}^2, P_f) and \mathcal{M}_f be the corresponding moduli space. We write

$\tau = \langle h \rangle$ if a point τ is represented by a diffeomorphism h . Correspondingly, points of \mathcal{M}_f are represented by $h(P_f)$ modulo post-composition with Moebius transformations. Denote by $\pi : \mathcal{T}_f \rightarrow \mathcal{M}_f$ the canonical covering map which sends $h \mapsto h|_{P_f}$. The (pure) mapping class group of (\mathbb{S}^2, P_f) is canonically identified with the group of deck transformations of π . For more background on Teichmüller spaces see, for example, [IT92, Hub06].

Consider an essential simple closed curve γ in (\mathbb{S}^2, P_f) . For each complex structure τ on (\mathbb{S}^2, P_f) , there exists a unique geodesic γ_τ in the homotopy class of γ . We denote by $l(\gamma, \tau)$ the length of the geodesic γ_τ homotopic to γ on the Riemann surface corresponding to $\tau \in \mathcal{T}_f$. This defines a continuous function from \mathcal{T}_f to \mathbb{R}_+ for any given γ . Moreover, $\log l(\gamma, \tau)$ is a Lipschitz function with Lipschitz constant 1 with respect to the Teichmüller metric (see [Hub06, Theorem 7.6.4]; note that in [DH93, Proposition 7.2] the constant is 2 because of a different normalization of the Teichmüller metric). We will use the same notation $l(\gamma, R)$ for the hyperbolic length of a curve γ in a hyperbolic surface R . Recall that, by the Collar Lemma, the length of a simple closed geodesic γ on a hyperbolic Riemann surface R is closely related to the supremum $M(\gamma, R)$ of moduli of all annuli on R that are homotopic to this geodesic, namely

$$\frac{\pi}{l(\gamma, R)} - 1 < M(\gamma, R) < \frac{\pi}{l(\gamma, R)}.$$

The Thurston's pullback σ_f is defined as follows. Suppose $\tau \in \mathcal{T}_f$ is represented by a homeomorphism h_τ . Consider the following diagram:

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & & \\ \downarrow f & & \\ (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f)) \end{array} \quad (2)$$

We can pull back the standard complex structure μ_0 on \mathbb{P} to an almost-complex structure $f^*h_\tau^*\mu_0$ on (\mathbb{S}^2, P_f) . By the Measurable Riemann mapping theorem, it induces a complex structure on (\mathbb{S}^2, P_f) . Let h_1 be a conformal isomorphism between (\mathbb{S}^2, P_f) endowed with the complex structure $f^*h_\tau^*\mu_0$ and \mathbb{P} . Set $\sigma_f(\tau) = \tau_1$ where τ_1 is the point represented by h_1 .

Now we can complete the previous diagram by setting $f_\tau = h_\tau \circ f \circ h_1^{-1}$ so that it commutes:

$$\begin{array}{ccc}
 (\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (\mathbb{P}, h_1(P_f)) \\
 \downarrow f & & \downarrow f_\tau \\
 (\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} & (\mathbb{P}, h_\tau(P_f))
 \end{array} \tag{3}$$

Note that from definition of f_τ , it follows that f_τ respects the standard complex structure μ_0 and, hence, is rational. When we choose a representing homeomorphism h_τ , we have the freedom to post-compose h_τ with any Moebius transformation; similarly, the choice of h_τ defines h_1 up to a post-composition by Moebius transformation. Thus, f_τ is defined up to pre- and post-composition by Moebius transformations.

The following proposition [DH93, Proposition 2.3] relates dynamical properties of σ_f to the original question.

Proposition 3.1. *A Thurston map f is equivalent to a rational function if and only if σ_f has a fixed point.*

The *canonical* obstruction Γ_f is the set of all homotopy classes of curves γ that satisfy $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$ for all (or, equivalently, for some) $\tau \in \mathcal{T}_f$. In [Sel11a] we proved the following (see also [Pil01]).

Proposition 3.2. *If Γ_f is not empty then it is a simple completely invariant Thurston obstruction.*

The following is immediate.

Proposition 3.3. $\Gamma_f = \Gamma_{f^r}$.

Proof. Take any $\tau \in \mathcal{T}_f$. If $\gamma \in \Gamma_f$ then $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$. In particular, $l(\gamma, \sigma_f^{nr}(\tau)) = l(\gamma, \sigma_{f^r}^n(\tau)) \rightarrow 0$, hence $\gamma \in \Gamma_{f^r}$.

Set $D = d_T(\tau, \sigma_f(\tau))$. Since σ_f is weakly contracting and $\log l(\cdot, \gamma)$ is 1-Lipschitz, we have $l(\gamma, \sigma_f^{nr+s}(\tau)) \leq e^{Ds} l(\gamma, \sigma_f^{nr}(\tau))$. Therefore, if $l(\gamma, \sigma_f^{nr}(\tau)) \rightarrow 0$ then $l(\gamma, \sigma_f^n(\tau)) \rightarrow 0$ as well. \square

The next theorems are due to Kevin Pilgrim [Pil01].

Theorem 3.4 (Canonical Obstruction Theorem). *If for a Thurston map with hyperbolic orbifold, its canonical obstruction is empty then it is Thurston equivalent to a rational function. If the canonical obstruction is not empty then it is a Thurston obstruction.*

Theorem 3.5 (Curves Degenerate or Stay Bounded). *For any point $\tau \in \mathcal{T}_f$ there exists a bound $L = L(\tau, f) > 0$ such that for any essential simple closed curve $\gamma \notin \Gamma_f$ the inequality $l(\gamma, \sigma_f^n(\tau)) \geq L$ holds for all n .*

Recall that the augmented Teichmüller space $\overline{\mathcal{T}}(S)$ is the space of all stable Riemann surfaces with marked points *with nodes* of the same type as S (see, for example, [Wol03, Wol09]). The *type* of a noded surface is defined by its topological type (more precisely, by the topological type of a surface one obtains by opening up all nodes) and the number of marked points (excluding nodes). The augmented Teichmüller space $\overline{\mathcal{T}}_f$ is a stratified space with strata corresponding to multicurves on (\mathbb{S}^2, P_f) . We denote by \mathcal{S}_Γ the stratum corresponding to the multicurve Γ , i.e., the set of all noded surfaces for which the nodes come from pinching all elements of Γ and there are no other nodes. In particular, $\mathcal{T}_f = \mathcal{S}_\emptyset$. Strata of $\overline{\mathcal{M}}_f$ are labeled by equivalence classes $[\Gamma]$ of multicurves, where two multicurves Γ_1 and Γ_2 are in the same class if and only if one can be transformed to the other by an element of the pure mapping class group or, equivalently, if the respective elements of Γ_1 and Γ_2 separate points of P_f in the same way.

In [Sel11b], the author proves the following statement.

Theorem 3.6. *Thurston's pullback map extends continuously to a self-map of the augmented Teichmüller space.*

Remark 3.7. Theorems 3.4 and 3.5 can be reformulated as follows. Note that this version does not require that Thurston map f has hyperbolic orbifold.

Theorem 3.8. *The accumulation set of $\pi(\sigma_f^n(\tau))$ in the compactified moduli space $\overline{\mathcal{M}}_f$ is a compact subset of $\mathcal{S}_{[\Gamma_f]}$. If Γ_f is not empty then the sequence $\sigma_f^n(\tau)$ tends to \mathcal{S}_{Γ_f} .*

We will need the following proposition (compare [DH93, Lemma 5.2]).

Proposition 3.9. *There exists an intermediate cover \mathcal{M}'_f of \mathcal{M}_f (so that $\mathcal{T}_f \xrightarrow{\pi_1} \mathcal{M}'_f \xrightarrow{\pi_2} \mathcal{M}_f$ are covers and $\pi_2 \circ \pi_1 = \pi$) such that*

- i. π_2 is finite,*

ii. the diagram

$$\begin{array}{ccc}
 \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\
 \pi \downarrow & \searrow \pi_1 & \downarrow \pi \\
 & \mathcal{M}'_f & \\
 \pi_2 \swarrow & & \searrow \tilde{\sigma}_f \\
 \mathcal{M}_f & & \mathcal{M}_f
 \end{array} \tag{4}$$

commutes for some map $\tilde{\sigma}_f: \mathcal{M}'_f \rightarrow \mathcal{M}_f$,

iii. If $\pi_1(\tau_1) = \pi_1(\tau_2)$ then $f_{\tau_1} = f_{\tau_2}$ up to pre- and post-composition by Moebius transformations.

Note that \mathcal{M}'_f is a quotient of \mathcal{T}_f by a subgroup G of the pure mapping class group of finite index. Then the quotient $\overline{\mathcal{M}}'_f$ of $\overline{\mathcal{T}}_f$ by the same subgroup will be a compactification of \mathcal{M}'_f . The covers π_1 and π_2 can be extended to the corresponding augmented spaces so that $\pi_2 \circ \pi_1 = \pi$ still holds. As in the case of the compactified moduli space, we parametrize boundary strata by equivalence classes of multicurves, two classes of simplified closed curves being equivalent if one can be mapped to the other by the action of G . Evidently, the whole diagram above also extends to the augmented spaces.

4 Thurston maps with parabolic orbifolds

A complete classification of postcritically finite branched covers (i.e. Thurston maps (f, P_f) , where P_f is the postcritical set of f) with parabolic orbifolds has been given in [DH93]. All rational functions that are postcritically finite branched covers with parabolic orbifold has been extensively described in [Mil06b]. However, no classification has been developed yet for general Thurston maps. In this section, we remind the reader of basic results on Thurston maps with parabolic orbifolds.

Recall that a map $f: (S_1, v_1) \rightarrow (S_2, v_2)$ is a covering map of orbifolds if $v_1(x) \deg_x f = v_2(f(x))$ for any $x \in S_1$. The following proposition from [DH93] is crucial.

Proposition 4.1. *i. If $f: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a postcritically finite branched cover, then $\chi(O_f) \leq 0$.*

ii. If $\chi(O_f) = 0$, then $f: O_f \rightarrow O_f$ is a covering map of orbifolds.

Equation (1) gives six possibilities for $\chi(O_f) = 0$. If we record all the values of v_f that are bigger than 1, we get one of the following orbifold signatures.

- (∞, ∞) ,
- $(2, 2, \infty)$,
- $(2, 4, 4)$,
- $(2, 3, 6)$,
- $(3, 3, 3)$,
- $(2, 2, 2, 2)$.

We are mostly interested in the last case. A $(2, 2, 2, 2)$ -map is a Thurston map that has orbifold with signature $(2, 2, 2, 2)$. From now on in this section, we always assume that f is a $(2, 2, 2, 2)$ -map. An orbifold with signature $(2, 2, 2, 2)$ is a quotient of a torus T by an involution i ; the four fixed points of the involution i correspond to the points with ramification weight 2 on the orbifold. Denote by p the corresponding branched cover from T to \mathbb{S}^2 ; it has exactly 4 simple critical points which are the fixed points of i . It follows that f can be lifted to a covering self-map \hat{f} of T (see [DH93]).

Take any simple closed curve γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$. Then $p^{-1}(\gamma)$ has either one or two components that are simple closed curves.

Proposition 4.2. *If there are exactly two postcritical points of f in each complementary component of γ , then the p -preimage of γ consists of two components that are homotopic in T and non-trivial in $H_1(T, \mathbb{Z})$. Otherwise, all preimages of γ are trivial.*

Proof. Note that postcritical points of f are, by definition, the fixed points of involution i and critical values of p . Since γ separates \mathbb{S}^2 into two connected components, if γ has exactly one preimage α , then α must separate T into two components and, thus, be contractible. The disk bounded by α in T is mapped to one of the connected components of the complement of γ with degree two. Therefore, there exists exactly one critical point of p in this disk, and, hence, exactly one postcritical point in its image.

If γ has two preimages, then they do not intersect, which implies that they are homotopic to each other or at least one of them is contractible. However, if one of the preimages is contractible, the other one also is, since they are mapped to each other by i . In this case, the disk bounded by a contractible preimage component is mapped by f one-to-one to a component of the complement of γ . Thus, this complementary component has no postcritical points.

The last case is when γ has two preimages that are homotopic to each other and are not trivial in $H_1(T, \mathbb{Z})$. Then the complement of the full preimage of γ has two components that are annuli that are mapped by p to complementary components of γ with degree two. It follows that there are exactly two postcritical points in each of these components. \square

Every homotopy class of simple closed curves γ on T defines, up to a sign, an element $\langle \gamma \rangle$ of $H_1(T, \mathbb{Z})$. If a simple closed curve γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$ has two p -preimages, then they are homotopic by the previous proposition. Therefore, every homotopy class of simple closed curves γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$ also defines, up to a sign, an element $\langle \gamma \rangle$ of $H_1(T, \mathbb{Z})$. It is clear that for any $h \in H_1(T, \mathbb{Z})$ there exists a homotopy class of simple closed curves γ such that $h = n\langle \gamma \rangle$ for some $n \in \mathbb{Z}$.

Since $H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$, the push-forward operator \hat{f}_* is a linear operator. It is easy to see that the determinant of \hat{f}_* is equal to the degree of \hat{f} , which is in turn equal to the degree of f . Existence of invariant multicurves for f is related to the action of \hat{f}_* on $H_1(T, \mathbb{Z})$.

Proposition 4.3. *Suppose that a component γ' of the f -preimage of a simple closed curve γ on $\mathbb{S}^2 \setminus \mathcal{P}_f$ is homotopic to γ . Take a p -preimage α of γ . Then $\hat{f}_*(\langle \alpha \rangle) = \pm d\langle \alpha \rangle$, where d is the degree of f restricted to γ' .*

Proof. By the previous proposition, if γ is not essential in $\mathbb{S}^2 \setminus \mathcal{P}_f$ then $\langle \alpha \rangle = 0$ and the claim holds. Otherwise, both γ and γ' have exactly two components in their p -preimages. Denote by α' a component of the p -preimage of γ' that is mapped to α by \hat{f} with degree d . A homotopy between γ and γ' lifts to a homotopy between α and α' on T . Therefore, α and α' define the same (up to a sign) element of $H_1(T, \mathbb{Z})$, i.e. $\langle \alpha' \rangle = \pm \langle \alpha \rangle$. Since $\hat{f}(\alpha') = \alpha$, we get that $\hat{f}_*(\langle \alpha \rangle) = \hat{f}_*(\pm \langle \alpha' \rangle) = \pm d\langle \alpha \rangle$. \square

More generally, we obtain the following.

Proposition 4.4. *Let γ be a simple closed curve on $\mathbb{S}^2 \setminus \mathcal{P}_f$ such that there are two points of the postcritical set \mathcal{P}_f in each complementary component of γ . If all components of the f -preimage of γ have zero intersection number with γ in $\mathbb{S}^2 \setminus \mathcal{P}_f$, then $\hat{f}_*(\langle \gamma \rangle) = \pm d \langle \gamma \rangle$, where d is the degree of f restricted to any preimage of γ .*

Proof. Let γ' be a component of the f -preimage of γ . As before, denote by α' a component of the p -preimage of γ' that is mapped by \hat{f} to a component α of the p -preimage of γ with some degree d . Then $\hat{f}_*(\langle \alpha' \rangle) = \pm d \langle \alpha \rangle$. Since γ and γ' have zero intersection number in $\mathbb{S}^2 \setminus \mathcal{P}_f$, the homotopy classes of α and α' in T also have zero intersection number. By Proposition 4.2, $\langle \alpha \rangle$ is non-zero in $H_1(T, \mathbb{Z})$; the last equality implies that $\langle \alpha' \rangle$ is also non-zero in $H_1(T, \mathbb{Z})$. It follows that $\langle \alpha' \rangle = \pm \langle \alpha \rangle$. \square

5 Topological characterization of canonical obstructions

Recall that the canonical obstruction of a Thurston map f is defined as the set of all homotopy classes of curves for which the length of the corresponding geodesics tends to 0 as we iterate in the Teichmüller space T_f . Notice that this definition makes sense in the case of Thurston maps with parabolic orbifolds. However, only one of the implications in Theorem 3.4 is true in this setting.

Proposition 5.1. *If a Thurston map f with a parabolic orbifold is Thurston equivalent to a rational map then its canonical obstruction Γ_f is empty.*

Proof. If we start iterating σ_f at a fixed point τ then, obviously, the lengths of all geodesics are uniformly bounded from below. \square

The other direction of Theorem 3.4 tells that if for a Thurston map with hyperbolic orbifold the canonical obstruction is empty then there exist no obstructions at all for this map. In the general case, the following is true.

Theorem 5.2. *Suppose that the canonical obstruction of a Thurston map f is empty, and Γ is a simple Thurston obstruction for f . Then*

f is a $(2, 2, 2, 2)$ -map and every curve of Γ has two postcritical points of *f* in each complementary component.

Proof. The map *f* must have parabolic orbifold by Theorem 3.4. Since the canonical obstruction is empty, the leading eigenvalue of M_Γ is equal to 1.

Case I. As an illustration of the idea of the proof, we first consider the case when Γ consists of a single simple closed curve γ . Then the Thurston matrix has the form $M_\Gamma = (1)$. Denote by $r(\tau) = M(\gamma, \tau)$ the maximal modulus of an annulus homotopic to γ in the Riemann surface corresponding to τ . (We can always find an annulus of maximal modulus by Lemma A.1.) Recall, that by the Collar Lemma, this is approximately equal to $1/l(\gamma, \tau)$. Then, by the Grötzsch inequality (see more detailed explanation below), $r(\sigma_f(\tau)) \geq r(\tau)$. As usual, denote $\tau_n = \sigma_f^n(\tau_0)$. The sequence $r_n = r(\tau_n)$ is increasing and bounded because γ is not a part of canonical obstruction and, therefore, $l(\gamma, \tau_n) > L > 0$ for some L . It follows that r_n has a limit which we denote by r .

We pick initial τ_0 so that $r_0 > \pi/\ln(3 + 2\sqrt{2})$ to make sure that the geodesic corresponding to γ is shorter than any simple closed geodesic that intersects γ by the Collar Lemma.

Theorem 3.8 implies that all $m'_n = \pi_1(\tau_n)$ belong to a compact subset of \mathcal{M}'_f , where π_1 is defined as before (see Proposition 3.9). Consider a subsequence n_k such that m'_{n_k} converges to $p \in \mathcal{M}'_f$ and m'_{n_k+1} converges to $q \in \mathcal{M}'_f$. On either of Riemann surfaces corresponding to p and q , there exists exactly one homotopy class of simple closed curves γ' such that γ' and γ are in the same equivalence class under the action of the covering group corresponding to π_1 , and that $M(\gamma', p) = r$ (or $M(\gamma', q) = r$ respectively). Indeed, any two different curves in the same equivalence class have non-zero intersection number because otherwise they would bound an annulus that cannot contain any of the marked points and, thus, be homotopic to each other. Since $r > r_0$ is large enough, $M(\gamma'', p) < r$ (and the same for q) for any other curve γ'' in the equivalence class of γ because otherwise γ' and γ'' would not intersect by the Collar Lemma. The same statement obviously holds for equivalence classes of curves with respect to the mapping class group.

Recall the commutative diagram (4):

$$\begin{array}{ccc} \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\ \pi_1 \downarrow & & \downarrow \pi \\ \mathcal{M}'_f & \xrightarrow{\tilde{\sigma}_f} & \mathcal{M}_f \end{array}$$

Since all the maps involved are continuous, it follows that $\tilde{\sigma}_f(p) = \lim \tilde{\sigma}_f(m'_{n_k}) = \lim \pi(\tau_{n_k+1}) = \pi_2(q)$. Take any point $\tau \in \mathcal{T}_f$ such that $\pi_1(\tau) = p$ and $r(\tau) = M(\gamma, \tau) = r$ (a point p provides a conformal structure on the base topological surface together with marking of some equivalence classes of curves, to get a point in the fiber $\pi_1^{-1}(p)$ we simply choose a representative of each marked class; for the equivalence class of γ we choose the shortest representative). Then from the same diagram we get $\pi(\sigma_f(\tau)) = \pi_2(q)$. Therefore $r(\sigma_f(\tau))$ is bounded above by $M(\gamma', \pi_2(q))$ where γ' is equivalent to γ under the action of the mapping class group. As we have seen above, this implies that $r(\sigma_f(\tau)) \leq r$. On the other hand, we know that $r(\sigma_f(\tau)) \geq r(\tau) = r$ which implies $r(\sigma_f(\tau)) = r(\tau) = r$. We now investigate under which conditions the inequality $r(\sigma_f(\tau)) \geq r(\tau)$ can become an equality.

Denote by A an annulus in the Riemann surface corresponding to τ that is homotopic to γ and has maximal possible modulus r (see Lemma A.1); let annuli A_1, \dots, A_k be the disjoint preimages of A in the Riemann surface corresponding to $\sigma_f(\tau)$ under the map f_τ that are homotopic to γ . Each A_i is mapped to A by f_τ as a non-ramified cover of degree d_i . Therefore, $\text{mod } A_i = 1/d_i \text{ mod } A = r/d_i$. By the definition of Thurston obstructions, $M_{\{\gamma\}} = (\sum 1/d_i)$ hence $\sum 1/d_i = 1$. Consider the annulus B containing all A_i and homotopic to γ . By the Grötzsch inequality, $\text{mod } B \geq \sum \text{mod } A_i = \sum (1/d_i)r = r$.

If we assume that $r(\sigma_f(\tau)) = r(\tau)$ then $\text{mod } B \leq r$, hence B is an annulus homotopic to γ with maximal possible modulus and the last inequality is, in fact, an equality. Hence, the closure of B must be the whole Riemann sphere; and A_i are round subannuli of B and their closure covers B . If there were only one preimage A_1 of A , then the degree d_1 would have to be equal to 1; this would imply that f is not a Thurston map but a homeomorphism because the closure of $B = A_1$ is the whole sphere. Every two adjacent annuli share a

common boundary component α which is an analytic curve. Then the corresponding boundary component $\beta = f_\tau(\alpha)$ of B is piecewise (except for possibly at the images of critical points) smooth. Since both adjacent to α annuli are mapped to A , β has only one complementary component. We conclude that β is a smooth curve segment connecting two critical values of f and passing through no other critical value, because a component of the preimage of β is a simple closed curve. Moreover, all critical points of f_τ in α are of degree 2. If the number of annuli k is at least 3 then the same is true for the other boundary component δ of A . Hence there exist at least 4 critical values – at least two in each complementary component of A – and this yields the statement of the theorem. We can reduce the case $k = 2$ to the case, say, $k = 4$ by considering the second iterate of f . Alternatively, we can prove it directly as follows.

The number of critical points on α is equal to twice the degree of f on either of the annuli; in particular, this yields d_i are all equal. Thus, in the case of exactly two annuli, the map f has degree 4. We already know that there are exactly 4 critical points of f_τ on α . Each of the two preimages δ_1 and δ_2 of δ must contain at least one critical point because otherwise $A_1 \cup \delta_1$ (or $A_2 \cup \delta_2$ respectively) contains no critical points and, hence, maps by f_τ conformally on $A \cup \delta$ which is a contradiction. Therefore, f has exactly 6 simple critical points since a branched cover of degree 4 has at most $4 * 2 - 2 = 6$ critical points. This implies that f is a $(2, 2, 2, 2)$ -map. Indeed, the degree of f is 4 so every critical value is the image of at most two different critical points. Thus, we must have at least 3 critical values and their full preimage consists of the 6 critical points. If the postcritical set contains only 3 points, then all critical values are also critical points and the signature of the orbifold corresponding to f is (∞, ∞, ∞) , which contradicts the fact that f has parabolic orbifold. We see that \mathcal{P}_f contains at least 4 points; since f has parabolic orbifold we conclude that f is a $(2, 2, 2, 2)$ -map.

Note that the endpoints of β are the only two postcritical points of f on β , because every other point on β has exactly 4 preimages on α and none of them are either critical or marked. Therefore, each complementary component of γ has two postcritical points.

Case II. We use the same approach in the case when $M_\Gamma = \{\gamma_1, \dots, \gamma_k\}$ is positive and $k \geq 2$. By the Perron-Frobenius theorem (Theorem B.1), there exists a positive eigenvector v corresponding

to $\lambda_\Gamma = 1$. Denote $r(\tau) = \sup \min_{i=\overline{1,k}} \text{mod } A_i/v_i$, where the supremum is taken over all configurations of disjoint annuli A_i in the Riemann surface corresponding to τ , such that A_i is homotopic to γ_i for $i = \overline{1,k}$.

Consider annuli B_i in the Riemann surface corresponding to $\sigma_f(\tau)$, such that B_i is homotopic to γ_i and B_i contains all preimages of A_j that are homotopic to γ_i for every $i = \overline{1,k}$. The same arguments as above show that $(\text{mod } B_i)^T \geq M_\Gamma (\text{mod } A_i)^T$. If A_i is the maximal configuration that realizes $r(\tau)$ (again, existence of such a configuration follows from Lemma A.1), then, by definition, $(\text{mod } A_i)^T \geq r(\tau)v$ and, hence, $(\text{mod } B_i)^T \geq M_\Gamma r(\tau)v = r(\tau)v$. Thus, $r(\sigma_f(\tau)) \geq r(\tau)$. We use pre-compactness of m'_n as above to construct a point τ such that $r(\sigma_f^2(\tau)) = r(\sigma_f(\tau)) = r(\tau) = r$. This means that there exists i for which $\text{mod } B_i = rv_i$ and the inequality corresponding to the i -th line of $(\text{mod } B_i)^T \geq M_\Gamma (\text{mod } A_i)^T$ is an equality. Since all entries of M_Γ are positive, this immediately forces $\text{mod } A_i = rv_i$ for all i . Because $r(\sigma_f^2(\tau)) = r$, the same reasoning implies that $\text{mod } B_i = rv_i$, for all i , and $\{B_i\}$ is a maximal configuration for $\sigma_f(\tau)$. In particular, we infer that the closure of the union of B_i covers the whole sphere, and Γ is invariant and no curve of Γ has a non-essential preimage.

Moreover, all f_τ -preimages of annuli A_j that are homotopic to γ_i are round subannuli of B_i and closure of their union covers B_i . If two preimages of some A_j abut along a boundary component then the image boundary component of A_j is a smooth curve segment (see above). If preimages of two different A_j and A_l abut in $\sigma_f(\tau)$ then the annuli themselves abut in τ . Since for every j , there exists at least one preimage of A_j that is homotopic to γ_i , all annuli are concentric and so are the curves γ_i . Since B_i are homotopic to γ_i , it follows that all preimages of A_i are also concentric. We see that no boundary component that separates two annuli A_i can contain a postcritical point. By removing marked points on these boundary components, we can reduce the statement of the theorem to the previous case.

Case III. The general case is now easily reduced to the previous case. Suppose Γ is an arbitrary simple obstruction. Take a subset Γ_1 such that M_{Γ_1} is irreducible and $\lambda_{\Gamma_1} = 1$ (see Section B). By Theorem B.5, some iterate k of M_{Γ_1} can be conjugated by a permutation matrix into the block form where all the blocks on diagonal

are positive and all other blocks are zero, moreover, the leading eigenvalue of each block is 1. Take further subset $\Gamma_2 \subset \Gamma_1$ corresponding to any of the diagonal blocks. Then M_{Γ_2} with respect to f^k is a positive matrix with leading eigenvalue 1. From the previous cases, we know that f^k is a $(2, 2, 2, 2)$ -map and that every element of Γ_2 has two postcritical points in each complementary component. This immediately implies, that f itself is a $(2, 2, 2, 2)$ -map with the same postcritical set. Moreover, we get that Γ_2 is invariant. Therefore, Γ_1 is a union of invariant multicurves for f^k and no curve of these multicurves has a non-essential preimage. It follows that Γ_1 is f -invariant and all curves in Γ_1 have two postcritical points in each complementary component.

Set $\Gamma' = \Gamma \setminus \Gamma_1$. Then $\lambda_{\Gamma'} = 1$ because Γ is simple and Γ_1 is invariant. We repeat the argument until we exhaust Γ , proving the desired property for all curves in Γ . \square

Recall that to a $(2, 2, 2, 2)$ -map f we associate a linear operator $\hat{f}_*: H_1(T, \mathbb{Z}) \rightarrow H_1(T, \mathbb{Z})$ (see Section 4). The following statement is in a certain sense a converse to the previous theorem.

Theorem 5.3. *The canonical obstruction of a $(2, 2, 2, 2)$ -map f contains a curve that has two postcritical points in each complementary component if and only if \hat{f}_* has two different integer eigenvalues.*

Proof. Let \hat{f}_* have two different integer eigenvalues d_1 and d_2 . Without loss of generality, we assume that both are positive and $d_1 < d_2$. We first assume that P_f has exactly four points. Take a curve γ in $\mathbb{S}^2 \setminus P_f$ such that $\langle \gamma \rangle$ is an eigenvector of \hat{f}_* corresponding to d_1 . For any connected component γ' of the f -preimage of γ , we have $\hat{f}_*(\langle \gamma' \rangle) = d \langle \gamma \rangle$ with some $d \in \mathbb{Z}$. It follows that $\langle \gamma' \rangle$ is also an eigenvector of \hat{f}_* corresponding to d_1 because \hat{f}_* is diagonalizable. We infer that all preimages of γ are homotopic to γ and each of them is mapped to γ by f with degree d_1 . Since the degree of f is equal to $d_1 d_2$, there are exactly d_2 preimages of γ . Therefore, $\Gamma = \{\gamma\}$ is an obstruction with $M_\Gamma = (d_2/d_1)$ and $\Gamma \subset \Gamma_f$ because the leading eigenvalue $\lambda_\Gamma = d_2/d_1 > 1$.

If P_f has more than four points, we first consider the Thurston map $F = (f, \mathcal{P}_f)$ where \mathcal{P}_f is the postcritical set of f . By the arguments above, there exists a curve γ that has two postcritical points in each complementary component which is a part of the canonical

obstruction of F . Clearly, there exists a canonical projection p from \mathcal{T}_f to \mathcal{T}_F that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{T}_f & \xrightarrow{\sigma_f} & \mathcal{T}_f \\ p \downarrow & & p \downarrow \\ \mathcal{T}_F & \xrightarrow{\sigma_F} & \mathcal{T}_F \end{array}$$

As we iterate $p(\tau_1)$ in the Teichmüller space \mathcal{T}_F , the maximal modulus $M(\gamma, p(\tau_m))$ of an annulus homotopic to γ tends to infinity. This is the same as the maximal modulus on an annulus A homotopic to γ in $\tau_m \in T_f$ if we fill in all extra punctures. The extra marked points split A into at most $k+1$ concentric annuli, where k is the number of points in $P_f \setminus \mathcal{P}_f$. Thus, the lengths of all homotopy classes of curves in $\mathbb{S}^2 \setminus P_f$ that are homotopic to γ relative \mathcal{P}_f can not be uniformly bounded from below as we iterate. Indeed, in this case all concentric annuli must have modulus uniformly bounded from above, and since there are at most $k+1$ of them, the modulus of an annulus homotopic to γ relative \mathcal{P}_f would be also bounded from above, which is a contradiction. By Theorem 3.8, there exists a curve homotopic to γ relative P'_f in the canonical obstruction Γ_f .

Let us now prove the converse direction; suppose that γ is a curve in the canonical obstruction of f that has two postcritical points in each complementary component. By Proposition 3.2, the canonical obstruction Γ_f is invariant. This implies that every component of the f -preimage of γ has zero intersection number with γ . Therefore, $\langle \gamma \rangle$ is an eigenvector of \hat{f}_* corresponding to some eigenvalue $d \in \mathbb{Z}$ by Proposition 4.4. If we identify $H_1(T, \mathbb{Z})$ with \mathbb{Z}^2 , then \hat{f}_* becomes a multiplication by an integer 2×2 matrix. We see that the other eigenvalue a of \hat{f}_* is also an integer. We suppose, by contradiction, that the other eigenvalue of \hat{f}_* is also equal to d . We fix a basis of $H_1(T, \mathbb{Z})$ containing $\langle \gamma \rangle$, the operator \hat{f}_* then assumes the form

$$\hat{f}_* = \begin{pmatrix} d & a \\ 0 & d \end{pmatrix},$$

where $a \in \mathbb{Z}$. As above, we consider the Thurston map $F = (f, \mathcal{P}_f)$ instead of f . Then γ (viewed as a curve on $\mathbb{S}^2 \setminus \mathcal{P}_f$) must lie in the canonical obstruction of F .

If $a = 0$, then \hat{f}_* is just a scalar multiplication by d so any element of $H_1(T, \mathbb{Z})$ is an eigenvector. Take any γ' such that γ'

is not homotopic to γ relative \mathcal{P}_f , and there are two postcritical points of f in each complementary component of γ' . Since $\langle \gamma' \rangle$ is an eigenvector of \hat{f}_* , we see that there exist d components of the preimage of γ' that are homotopic to γ' and are mapped to γ' with degree d . Thus, the multicurve consisting of a single curve γ' is an obstruction for F and, hence, γ' can not have positive intersection with any curve in the canonical obstruction of F , in particular, with γ . This is clearly a contradiction because \mathcal{P}_f has only four points.

Similar argument also works in the case $a \neq 0$. We may assume that $a = d \cdot b$ with $b \in \mathbb{Z}$, otherwise we can consider F^d , for which γ will be still in the canonical obstruction, but the action on $H_1(T, \mathbb{Z})$ will be given by

$$\hat{f}_*^d = \begin{pmatrix} d & a \\ 0 & d \end{pmatrix}^d = \begin{pmatrix} d^d & d^d a \\ 0 & d^d \end{pmatrix}.$$

Consider a curve γ_0 such that $\langle \gamma_0 \rangle = (0, 1)^T$. Then for a component γ_1 of the preimage of γ_0 we have $\hat{f}_*(\langle \gamma_1 \rangle) = d_0 \langle \gamma_0 \rangle = (0, d_0)^T$ for some $d_0 \in \mathbb{Z}$. The coordinates x, y of $\langle \gamma_1 \rangle = (x, y)^T$ must be co-prime since γ_1 is a simple closed curve. Therefore, from

$$\hat{f}_* \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & d \cdot b \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(x + by) \\ dy \end{pmatrix} = \begin{pmatrix} 0 \\ d_0 \end{pmatrix}$$

we infer that $y = 1$ and $d_0 = d$. (Recall that coordinates of $\langle \gamma_1 \rangle$ are defined up to a sign, depending on the chosen orientation; we can, hence, force y to be positive.) As above, it follows that γ_0 has exactly d preimages that are all homotopic to γ_1 and mapped to γ_1 with degree d , where $\langle \gamma_1 \rangle = (-b, 1)^T$. By induction, the same statement is true for all $i \in \mathbb{N}$ with $\langle \gamma_i \rangle = (-i \cdot b, 1)^T$.

The curves γ_i form an infinite analogue of a Thurston obstruction; the corresponding linear transformation M_Γ acts on $\mathbb{R}^\mathbb{N}$ by shifting the indexes of basis vectors. By the same reasoning as above, we get $M(\gamma_{i+1}, \tau_{i+1}) \geq M(\gamma_i, \tau_i)$ for all $i \in \mathbb{N}$. Since all γ_i have positive intersection with γ , we conclude that γ can not be in the canonical obstruction of F , which is a contradiction. \square

Corollary 5.4. *The canonical obstruction of a $(2, 2, 2, 2)$ -map f is empty if and only if every curve of every simple Thurston obstruction for f has two postcritical points of f in each complementary component and the two eigenvalues of \hat{f}_* are equal or non-integer.*

For any other Thurston map, the canonical obstruction is empty if and only if there exist no Thurston obstruction for f .

Proof. The statement follows directly from the previous two theorems. \square

The following theorem is a generalization of Pilgrim's conjecture ([Sel11b, Theorem 10.3]). We use a different approach here.

Theorem 5.5. *If the first-return map F of a periodic component C of the noded topological surface corresponding to pinching curves in Γ_f is a Thurston map then the canonical obstruction of F is empty.*

Proof. The main idea of the proof is essentially the same as in the proof of Theorem 5.2. Suppose, on contrary, that $\Gamma \neq \emptyset$ is the canonical obstruction of F . Passing to an appropriate iterate of f , we may assume that C is a fixed component (see Proposition 3.3). In this case, Γ is f -invariant. Since Γ is not a part of the canonical obstruction, the leading eigenvalue λ_Γ is equal to 1. Taking a higher iterate of f , if needed, we can find a subset $\Gamma' \subset \Gamma$ such that $\lambda_{\Gamma'} = 1$ and $M_{\Gamma'}$ is positive.

We can repeat the proof of Case II of Theorem 5.2 almost verbatim. As before, there exists a positive eigenvector v corresponding to the leading eigenvalue 1. Denote $r(\tau) = \sup \min_{i=1,k} \text{mod } A_i/v_i$, where the supremum is taken over all configurations of disjoint annuli A_i in the Riemann surface corresponding to τ , such that A_i is homotopic to $\gamma_i \in \Gamma'$. We have already established that $r(\sigma_f(\tau)) \geq r(\tau)$. Take an accumulation point m' of the projection of an arbitrary orbit to the space \mathcal{M}' . Then $m' \in \mathcal{S}_{\Gamma_f}$ by Theorem 3.8 and $r(m') = r(\tilde{\sigma}_f(m'))$. Since all annuli that are homotopic to curves in Γ' on the noded surface corresponding to any noded surface in \mathcal{S}_{Γ_f} must be contained in C , from this point on the proof goes the same way.

We conclude that F is a $(2, 2, 2, 2)$ -map and all curves of Γ' have two postcritical points of F in each complementary component. However, by the previous theorem, no such curve can be contained in the canonical obstruction of F . Otherwise, the leading eigenvalue of Γ' would be strictly greater than 1, which would force Γ' to be a part of the canonical obstruction of f . This contradiction shows that the canonical obstruction of F is empty. \square

We are now in the position to formulate and prove a pure topological criterion that singles out canonical obstructions. It says that the canonical obstruction is the minimal obstruction satisfying the conclusion of the previous theorem.

Theorem 5.6 (Characterization of Canonical Thurston Obstructions). *The canonical obstruction Γ is a unique minimal Thurston obstruction with the following properties.*

- *If the first-return map F of a cycle of components in \mathcal{S}_Γ is a $(2, 2, 2, 2)$ -map, then every curve of every simple Thurston obstruction for F has two postcritical points of f in each complementary component and the two eigenvalues of \hat{F}_* are equal or non-integer.*
- *If the first-return map F of a cycle of components in \mathcal{S}_Γ is not a $(2, 2, 2, 2)$ -map or a homeomorphism, then there exists no Thurston obstruction of F .*

Proof. By Corollary 5.4, both conditions above are equivalent to saying that the canonical obstruction for F is empty. The necessity of these conditions then follows from Theorem 5.5. Suppose, on contrary, that Γ is not minimal with these properties, i.e. there exists $\Gamma' \subsetneq \Gamma$ satisfying the same condition. Since Γ is simple by Proposition 3.2, at least one curve γ of $\Gamma \setminus \Gamma'$ must lie in a periodic component C of $\mathcal{S}_{\Gamma'}$. Consider the multicurve Γ_1 containing all curves of $\Gamma \setminus \Gamma'$ that lie in C ; as Γ is simple, Γ_1 is an obstruction for F . There are three cases.

Case I. The first-return map to C is a homeomorphism. Since γ is essential and not homotopic to any nodes, the component C must have at least four marked points. Take any other simple closed curve α_1 in C that has non-zero intersection with γ . Since the first-return map is a homeomorphism, γ is a part of a Levy cycle. Denote by α_2 the one-to-one preimage of α_1 and so on. Then, as in the proof of Theorem 5.3, we see that $M(\alpha_{i+1}, \tau_{i+1}) \geq M(\alpha_i, \tau_i)$ and α_{ki} intersects γ for all i , where k is the length of the Levy cycle. This implies that the length of γ is bounded from above, which contradicts the assumption that γ was a part of the canonical obstruction.

Case II. The first-return map to C is a $(2, 2, 2, 2)$ -map. It follows that every curve in Γ_1 has two postcritical points of F in each

complementary component. By the same argument as in the proof of Theorem 5.3, the length of these curves can not tend to zero as we iterate f so Γ_1 can not be a part of the canonical obstruction.

Case III. The first-return map to C is neither of the two cases above. By Corollary 5.4, F has no obstructions at all, hence Γ_1 must be empty.

Since any curve of any other obstruction either lies in Γ , or does not intersect any curve in Γ , the uniqueness of a minimal obstruction satisfying the conditions of the theorem follows from the same argument. \square

A Continuity of modulus

In this section we prove the following lemma (compare to [LV73, Section I.4.9]).

Lemma A.1. *Let A_i be an annulus in \mathbb{P} with two complementary components B_i and C_i , for every $i \in \mathbb{N}$. Suppose that $B_i \rightarrow B$ and $C_i \rightarrow C$ as $i \rightarrow \infty$ with respect to the Hausdorff metric, and both B and C contain at least two points. If there exists a doubly connected component A of $\mathbb{P} \setminus \{B \cup C\}$ then $\text{mod } A_i \rightarrow \text{mod } A$; otherwise $\text{mod } A_i \rightarrow 0$.*

Proof. First we note that B_i and C_i are connected for every i , therefore B and C are connected. This immediately implies that there exists at most one doubly connected component A in the complement of B and C .

Suppose A exists. We may choose a compactly contained in A and homotopic to A sub-annulus A' such that $\text{mod } A - \text{mod } A' = \varepsilon$ where $\varepsilon > 0$ is arbitrarily small. Then the two complementary components B' and C' of A' are compact neighborhoods of B and C respectively. For i large enough, we have that $B_i \subset B'$ and $C_i \subset C'$ and, hence, $A_i \supset A'$. We see that $\text{mod } A_i > \text{mod } A' = \text{mod } A - \varepsilon$. We conclude that $\liminf(\text{mod } A_i) \geq \text{mod } A$.

Let $b_1, b_2 \in B$ and $c \in C$ be three distinct points in \mathbb{P} . For i large enough, there exist three distinct points b_1^i, b_2^i and c^i such that $b_1^i, b_2^i \in B_i$ and $c^i \in C_i$, and $d(b_1, b_1^i) < \varepsilon$, $d(b_2, b_2^i) < \varepsilon$ and $d(c, c^i) < \varepsilon$. Letting ε go to 0, we construct a sequence $\{m_i\}$ of Moebius transformations such that $b_1, b_2 \in m_i(B_i)$, $c \in m_i(C_i)$ and $\{m_i\}$ converges uniformly to the identity map on \mathbb{P} . Then the Hausdorff

distance between B_i and $m_i(B_i)$ tends to 0 as $i \rightarrow \infty$, and $m_i(B_i) \rightarrow B$ and, analogously, $m_i(C_i) \rightarrow C$. Since $\text{mod } m_i(A_i) = \text{mod } A_i$, it is enough to prove the statement of the lemma for the sequence $\{m_i(A_i)\}$.

Thus, we may assume that $b_1, b_2 \in B_i$ and $c \in C_i$ for all i . Let $a = \limsup(\text{mod } A_i) > 0$; pick a sequence n_i such that $\text{mod } A_{n_i} > a - \varepsilon$ for all $i \in \mathbb{N}$. Consider a sequence of sub-annuli A'_{n_i} of A_{n_i} with $\text{mod } A'_{n_i} = a - \varepsilon$ for all $i \in \mathbb{N}$; let $\{f_i : R \rightarrow A'_{n_i}\}$ be a sequence of conformal isomorphisms from a round annulus R of modulus $a - \varepsilon$ onto A'_{n_i} . Since all f_i do not assume values b_1, b_2 or c on R , the family $\{f_i\}$ is normal by Montel's theorem and there exists a subsequence that converges locally uniformly to a holomorphic map f defined on R . Clearly the diameters of sets B_i tend to the diameter of B and the diameters of sets C_i tend to the diameter of C . The core curve of R is mapped by f_i to a smooth Jordan curve separating B_i and C_i and therefore the lower limit of the diameter of this curve is positive. This yields that f can not be constant and, hence, is a conformal map onto an annulus that separates b_1, b_2 and c . Let us prove that $f(R) \cap B = \emptyset$.

On contrary, let $f(z) = b \in B$ for some point $z \in R$. To simplify the notation, we assume that $b \neq \infty$ and that $f_i \rightarrow f$. Let r be the distance between z and the boundary of the annulus R . For any $\varepsilon > 0$, if i is large enough there exists a point $b'_i \in B_i$ such that $|b'_i - b| < \varepsilon$ and $|f_i(z) - b| < \varepsilon$. It follows that $|f_i(z) - b'_i| < 2\varepsilon$ but b'_i is not in the image of f_i . Koebe 1/4 theorem implies that $f'_i(z) < 8\varepsilon/r$. Hence, $f'(z) = \lim f'_i(z) = 0$, which contradicts the fact that f is conformal. Therefore $f(R)$ does not intersect B nor, by the same argument, C . We conclude that the component A of the complement of $B \cup C$ containing $f(R)$ is an annulus and $\text{mod } A \geq \text{mod } f(R) = \text{mod } R = a - \varepsilon$. We see that $\limsup(\text{mod } A_i) \leq \text{mod } A$. \square

B Positive matrices

This work uses classical results from matrix theory (see, for example, [Gan66]).

Definition B.1. We say that a matrix M is *non-negative* (or *positive*) and write $M \geq 0$ (or $M > 0$) if all its entries are non-negative (or, respectively, positive). Same definition also applies for vectors.

More generally, we write $A \geq B$ (or $A > B$) if $A - B \geq 0$ (respectively, $A - B > 0$).

Definition B.2. A square matrix M is *reducible* if there exists a permutation matrix P such that conjugation by P puts M in the block form

$$P^{-1}MP = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}, \quad (5)$$

where M_{11} and M_{22} are square matrices. If no such permutation exists, the matrix M is *irreducible*.

It is obvious that any positive matrix is irreducible. Conjugating by a permutation matrix, any matrix can be written in the form

$$P^{-1}MP = \begin{pmatrix} M_{11} & 0 & \dots & 0 \\ M_{21} & M_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ M_{k1} & M_{k2} & \dots & M_{kk} \end{pmatrix},$$

where all blocks M_{ii} are square and irreducible. Clearly, the spectral radius of M is the maximum of spectral radii of M_{kk} .

The following theorem has numerous implications in many subjects.

Theorem B.1 (Perron-Frobenius). *If M is an irreducible non-negative square matrix then there exist a unique largest eigenvalue (the Perron-Frobenius or leading eigenvalue) $\lambda(M)$ which is real and positive, and a unique up to scale positive eigenvector with eigenvalue $\lambda(M)$.*

Corollary B.2. *If M is a non-negative square matrix then there exist a largest eigenvalue $\lambda(M)$ which is real and positive.*

Proof. The statement follows immediately from the previous theorem and preceding remark. \square

Definition B.3. A matrix M is called *primitive* if there exists a power $k \in \mathbb{N}$ for which M^k is positive.

Denote by I_n the $n \times n$ identity matrix. The following is a well-known result.

Theorem B.3. *If M is an irreducible non-negative $n \times n$ matrix then $(I_n + M)^{n-1} > 0$. In particular, the matrix $I_n + M$ is primitive.*

We prove a slightly more general statement.

Proposition B.4. *If M is an irreducible non-negative $n \times n$ matrix and at least one diagonal entry of M is positive then $M^{2n-2} > 0$ and, hence, M is primitive.*

Proof. Without loss of generality, we can assume that all non-zero entries of M are equal to 1. Construct a directed graph G with n vertices using M as an adjacency matrix, i.e. adding an edge from i -th to j -th vertex if and only if the corresponding entry m_{ij} is equal to 1. Since M is irreducible, there exists a directed path in G between any two vertices. Indeed, take any vertex a and denote by A the set of all vertices you can reach starting at a . If A is not the whole set of vertices then a permutation, that puts all vertices in A before the rest of the vertices, conjugates M to a matrix in the block form (5). Note that the shortest path between any two vertices is evidently no longer than $n - 1$.

We write $M^k = (m_{ij}^k)$. We notice that m_{ij}^k is equal to the number of paths in G of length exactly k that start at i -th vertex and end at j -th vertex. Therefore, it is enough to prove that between any two points there exists a path of length $2n - 2$. Recall that M has a non-zero diagonal entry which corresponds to a loop in G at some vertex v . Any two vertices a and b can be connected by a path of length at most $2n - 2$ that passes through v because there exist paths of length at most $n - 1$ connecting a to v and v to b . To construct a path of length $2n - 2$ we simply insert the loop at v an appropriate number of times into the former path. \square

We will also use the following statement (compare to Section XIII.5 in [Gan66]).

Theorem B.5. *For any irreducible matrix M there exists a power $k \in \mathbb{N}$ such that M^k , conjugating by an appropriate permutation P can be written in the block diagonal form*

$$P^{-1}M^kP = \begin{pmatrix} M_{11} & 0 & \dots & 0 \\ 0 & M_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{kk} \end{pmatrix},$$

where all M_{ii} are positive and have the same leading eigenvalue as M^k .

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